Technical Report: Minimizing Clutter Using Absence in Venn-i^e

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We set up a series of definitions and lemmas in order to prove theorem 3: algorithm 1 produces a minimally cluttered diagram.

Definition 1. *Let D be a consistent Venn-i^e diagram. A* ***canonical interpretation****, $\mathcal{I} = (U, \psi, \Psi)$, for D is defined as follows:*

1. *The universal set is the set of non-shaded zones in D: $U = Z \backslash ShZ$.*
2. *We now define $\psi: \mathcal{C} \rightarrow U$. Let $\iota \in \mathcal{C}$. We identify three cases in order to specify $\psi(\iota)$.*
   (a) There exists an (r, ι) in D. *Since D is consistent, there is a zone, z, such that*
   $$z \in \bigcap_{(r',\iota) \in I(\iota)} r'$$
   where z is not shaded and $(z, \iota) \notin I(\bar{\iota})$. Choose any such z and define $\psi(\iota) = z$.
   (b) There is no (r, ι) in D but there is an (z, ι) in D. *Since D is consistent, there is a non-shaded zone, z' such that*
   $$z' \in Z \backslash (\{z'' : (z'', \iota) \in I(\bar{\iota})\} \cup ShZ).$$
   Choose any such z' and define $\psi(\iota) = z'$.
   (c) Otherwise, there is no (r, ι) in D and no (z, ι) in D. *Since D is consistent, there is a non-shaded zone, z, in U. Choose an arbitrary non-shaded zone, z, and define $\psi(\iota) = z$.*
3. *We define $\Psi: \mathcal{L} \rightarrow \mathbb{P}U$ by, for each label λ in $\mathcal{L}$,*
   $$\Psi(\lambda) = \{(in, out) \in U : \lambda \in in\}.$$

We will show that the canonical interpretation is a model for D. First, we establish the set to which each zone in D maps.

Lemma 1. *Let D be a consistent Venn-i^e diagram and let $\mathcal{I} = (U, \psi, \Psi)$ be a canonical interpretation for D. Then*

1. *for each zone, (in, out), in $Z \backslash ShZ$, $\Psi(in, out) = \{(in, out)\}$, and*
2. *for each zone, (in, out), in $ShZ \cup MZ$, $\Psi(in, out) = \emptyset$.*

Proof. Let (in, out) be a zone in $Z \cup ShZ \cup MZ$. Then

$$\begin{aligned}
\Psi(in, out) &= \bigcap_{\lambda \in in} \Psi(\lambda) \cap \bigcap_{\lambda \in out} (U \backslash \Psi(\lambda)) \\
&= \bigcap_{\lambda \in in} \{(in', out') \in U : \lambda \in in'\} \cap \bigcap_{\lambda \in out} (U \backslash \{(in', out') \in U : \lambda \in in'\}) \\
&= \{(in', out') \in U : in \subseteq in'\} \cap \bigcap_{\lambda \in out} \{(in', out') \in U : \lambda \notin in'\} \\
&= \{(in', out') \in U : in \subseteq in'\} \cap \bigcap_{\lambda \in out} \{(in', out') \in U : \lambda \in out'\} \\
&= \{(in', out') \in U : in \subseteq in'\} \cap \{(in', out') \in U : out \subseteq out'\}.
\end{aligned}$$

Suppose some zone, (in', out'), is in $\Psi(in, out)$. Then $in \subseteq in'$ and $out \subseteq out'$. But for any two zones, (X_1, Y_1) and (X_2, Y_2), in U we have $X_1 \cup Y_1 = X_2 \cup Y_2$. From this it follows that there is at most one zone, namely (in, out), in $\Psi(in, out)$. Moreover, this zone is in $\Psi(in, out)$ if and only if (in, out) is in U. Thus:

1. if $(in, out) \in Z \backslash ShZ = U$ then $\Psi(in, out) = \{(in, out)\}$, and
2. if $(in, out) \in ShZ \cup MZ$ and, therefore, not in U then $\Psi(in, out) = \emptyset$

as required □

Lemma 2. *Let D be a consistent Venn-i^e diagram and let $\mathcal{I} = (U, \psi, \Psi)$ be a canonical interpretation for D. Then $\mathcal{I}$ is a model for D.*

Proof. Having established the sets to which zones map in lemma 1, we now prove that the conditions for $\mathcal{I}$ to model D are satisfied.

1. *Missing Zones Condition.* Let $(in, out) \in MZ$. Then $\Psi(in, out) = \emptyset$, as required.
2. *Shaded Zones Condition.* Let $(in, out) \in ShZ$. Then $\Psi(in, out) = \emptyset$, as required.
3. $\otimes$*-Sequence Condition.* Let $(r, \otimes)$ be an $\otimes$-sequence in D. Then, since D is consistent, there is a non-shaded zone, z say, in r. In other words, $z \in Z \backslash ShZ = U$, so $\Psi(z) = \{z\} \neq \emptyset$, as required.
4. i*-Sequence Condition.* Let (r, ι) be an i-sequence in D. By the definition of ψ, there is a zone z where $\psi(\iota) = z$ and $z \in r$, implying $\psi(\iota) \in \Psi(z)$, where as required.
5. $\bar{i}$*-Sequence Condition.* Let (z, ι) be an $\bar{i}$-sequence in D. If there exists an (r, ι) in D then we defined $\psi(\iota) = z'$, for some zone z', where $(z', \iota) \notin I(\bar{\iota})$. This implies that $z' \neq z$ and, thus, $\psi(\iota) \notin \Psi(z)$, as required. Alternatively, there is no such (r, ι) in which case we defined $\psi(\iota)$ be some zone, z', not equal to z. Therefore, $\psi(\iota) = z' \notin \Psi(z)$, as required. In either case, the $\bar{i}$-sequence condition holds.

Hence, we have proved that $\mathcal{I}$ models D. □

Whilst not necessary for the proof of theorem 3, we can deduce that consistent diagrams are satisfiable, which is part of theorem 1:

Corollary 1. *Let D be a consistent Venn-i^e diagram. Then D is satisfiable.*

Returning our focus to proving theorem 3, one important property of minimal diagrams, and of the output of algorithm 1, is that any constant symbol, ι, is 'used' at most once for an i-sequence. Moreover, if ι is used for an i-sequence then it is not also used for an $\bar{i}$-sequence. This important property is captured in the following definition and allows us to construct canonical interpretations for minimal diagrams, and the output of algorithm 1, with specific properties.

Definition 2. *Let D be a consistent Venn-i^e diagram. If, for each constant, ι, in $\mathcal{C}$,*

1. *there at most one region, r, such that (r, ι) is in d, or*
2. *$I(\bar{\iota}) \neq \emptyset$,*

but not both then D ***represents its constants simply****.*

Lemma 3. *Let D be a minimally cluttered consistent Venn-i^e diagram. Then D represents its constants simply.*

Proof. Suppose that D does not represent its constants simply. Assume that there are two regions, r_1 and r_2, and a constant symbol, ι, such that (r_1, ι) and (r_2, ι) are in D. Then the set $I(\iota)$ is reducible in D and, therefore, D is not minimal. Alternatively, there is a region, r_1, a zone, z, and a constant symbol, ι, such that (r_1, ι) and (z, ι) are in D. Then we can swap $I(\bar{\iota})$ for an i-sequence, say (r_2, ι) using inference rule 2. After this swap $I(\iota)$ is reducible. Applying inference rule 4 results in the removal of (r_1, ι) and (r_2, ι) and the introduction of a new ι-sequence, namely $(r_1 \cap r_2, \iota)$; note here that r_1 cannot include shaded zones, since (r_1, ι) would then be reducible, and r_2 does not contain shaded zones since (r_2, ι) was introduced using inference rule 2. The resulting diagram differs from D in that it has no $\bar{\iota}$-sequences and (r_1, ι) became $(r_1 \cap r_2, \iota)$. The clutter score has reduced, because the $\bar{\iota}$-sequences have been removed and $|r_1 \cap r_2| \leq |r_1|$. This again contradicts the minimality of D. Hence D represents its constants simply. □

Lemma 4. *Let D be a consistent Venn-i^e diagram and let D_{min} be the output from algorithm 1, given the input D. Then D_{min} represents its constants simply.*

Proof. Step 1 of algorithm 1 removes all $\bar{i}$-sequences. For each ι, step 2 reduces all ι-sequences in $I(\iota)$ to a single ι-sequence. At this point, the diagram represents its constants simply. Trivially, steps 3 and 4 do not alter the simple representation of constants. Lastly, step 5 simply swaps single i-sequences for sets of $\bar{i}$-sequences, again not altering the simple representation of constants. Therefore, the output of algorithm 1, namely D_{min}, represents its constants simply. □

Lemma 5. *Let D be a consistent Venn-i^e diagram that represents its constants simply. Then, for each i-sequence, (r, ι), in D, and for each zone, z, in $r \backslash ShZ$, there exists a model, $\mathcal{I} = (U, \psi, \Psi)$, for D such that*

$$\psi(\iota) \in \Psi(z).$$

Proof. Let (r, ι) be an i-sequence in D and let z be a zone in $r \backslash ShZ$. We prove that some canonical interpretation, $\mathcal{I} = (U, \psi, \Psi)$, ensures $\psi(\iota) \in \Psi(z)$. Since D represents its constants simply we know that $I(\iota) = \{(r, \iota)\}$, so

$$z \in \bigcap_{(r'\iota) \in I(\iota)} r' = r.$$

We also know that the $\bar{i}$-sequence (z, ι) is not in D, because D represents its constants simply. Therefore, we can choose $\psi(\iota) = z$ when building a canonical interpretation. Thus, there exists a canonical interpretation, $\mathcal{I}$, where $\psi(\iota) = z$. Hence $\psi(\iota) \in \Psi(z)$, by lemma 1. By lemma 2, we know $\mathcal{I}$ models D. □

Lemma 6. *Let D be a consistent Venn-i^e diagram that represents it constants simply. Then, for each $\bar{i}$-sequence, (z, ι), in D, and for each zone, z, in $Z \backslash (\{z' : (z', \iota) \in I(\bar{\iota})\} \cup ShZ)$, there exists a model, $\mathcal{I} = (U, \psi, \Psi)$ for D such that*

$$\psi(\iota) \in \Psi(z).$$

Proof. Let (z, ι) be an $\bar{i}$-sequence in D and let z be a zone such that

$$z \in Z \backslash (\{z' : (z', \iota) \in I(\bar{\iota})\} \cup ShZ).$$

We prove that some canonical interpretation, $\mathcal{I} = (U, \psi, \Psi)$, ensures $\psi(\iota) \in \Psi(z)$. We know that there is no region, r, such that (r, ι) is in D, since D represents its constants simply. Therefore, we can choose $\psi(\iota) = z$ when building a canonical interpretation. Thus, there exists a canonical interpretation, $\mathcal{I}$, where $\psi(\iota) = z$. So, by lemma 1, $\psi(\iota) \in \Psi(z)$. By lemma 2, $\mathcal{I}$ models D. □

We can now prove that algorithm 1 minimizes clutter.

Theorem 3 (Clutter Minimization). *Let D be a consistent Venn-i^e diagram and let D_{min} be the result of applying algorithm 1 to D. Then D and D_{min} are semantically equivalent and D_{min} is minimally cluttered.*

Proof. The semantic equivalence of D and D_{min} follows from theorem 2. Suppose $D_{min} = (L, Z, ShZ, \rho_\otimes, \rho_i, \rho_{\bar{i}})$ is semantically equivalent to $D' = (L, Z, ShZ, \rho'_\otimes, \rho'_i, \rho'_{\bar{i}})$ and that D' is minimally cluttered. We use D' to show that D_{min} is minimally cluttered. In particular, we show that D' can be transformed into D_{min} without changing the clutter score. The proof breaks down into four main parts.
Part 1 The goal of part 1 is to establish that $\rho'_i \subseteq \rho_i$. Let (r', ι) be an i-sequence in D'. Then either:

(a) there is a region, r, such that (r, ι) is in D_{min},

(b) there is a zone, z, such that (z, ι) is in D_{min}, or
(c) neither of the above.

In case (a), we show that $r' = r$. By lemma 6, D' represents its constants simply. Therefore, by lemma 5, for each zone, z', in r' there is a model, $\mathcal{I}$, for D' such that

$$\psi(\iota) \in \Psi(z').$$

Then $\mathcal{I}$ is also a model for D_{min} because D_{min} is semantically equivalent to D'. By the i-sequence condition for D_{min}, we know that $\psi(\iota) \in \Psi(z)$ for some $z \in r$. Since distinct zones in Z represent disjoint sets, it follows that $z = z'$. Therefore $z' \in r$, so $r' \subseteq r$. Similarly, using lemmas 4 and 5, we can show that $r \subseteq r'$. Therefore $r' = r$, so (r', ι) is in D_{min}.

We now show that cases (b) and (c) cannot happen. For case (b), suppose there is a zone, z, such that (z, ι) in D_{min}. The only way such an $\bar{i}$-sequence can be in D_{min} is due to swapping an i-sequence, (r, ι), for the set of $\bar{i}$-sequences, $I(\bar{\iota})$, in the last step of algorithm 1, reducing the clutter score. In other words, we have

$$|Z \backslash (ShZ \cup r)| < 2|r| - 1.$$

Use inference rule 2 (swap $\bar{i}$-sequence) to swap the set $I(\bar{\iota})$ in D_{min} for (r, ι), to give a diagram we call D^s_{min}. This swap necessarily implies that the clutter score of D_{min} is lower than that of D^s_{min}. We also know that D^s_{min} represents its constants simply. But then similarly to case (a) we can show that $r' = r$. However, this implies that D' is not minimally cluttered: we could swap (r', ι) for the set $I(\bar{\iota})$ and reduce the clutter score. Therefore no such (z, ι) exists in D_{min}, that is case (b) cannot happen.

Focusing on case (c), we know that the region r' does not include all non-shaded zones, since this would imply that (r', ι) is redundant and D' would not be minimal. Choose a zone, z', that is not in r' and is not shaded. Then there is a canonical interpretation, $\mathcal{I}$, for D_{min} where $\psi(\iota) = z'$, noting case 2(c) of definition 1. We know that $\mathcal{I}$ models D_{min}, by lemma 2. However, $\mathcal{I}$ cannot model D', because the i-sequences condition is not satisfied. Since D_{min} and D' are semantically equivalent, we have reached a contradiction. Therefore, case (c) cannot happen. We deduce, since only case (a) can happen, that $\rho'_i \subseteq \rho_i$.

Part 2 The goal of part 2 is to establish that there is a minimally cluttered diagram, say

$$D'_2 = (L, Z, ShZ, \rho'_{\otimes 2}, \rho'_{i2}, \rho'_{\bar{i}2}),$$

where $\rho'_{i2} = \rho_i$. Let (r, ι) be an i-sequence in ρ_i that is not in ρ'_i. Then either:

(a) there is a region, r', such that (r', ι) is in D',
(b) there is a zone, z', such that (z', ι) is in D', or
(c) neither of the above.

We show that only case (b) can happen. Firstly, for case (a), we would also have that (r', ι) is in D_{min}, since $\rho'_i \subseteq \rho_i$. We know that $r \neq r'$, since (r, ι) is not in ρ'_i. But, since (r, ι) and (r', ι) are both in D_{min} this contradicts the fact that D_{min} represents its constants simply (lemma 4). Thus case (a) cannot happen.

For case (b), consider the set of $\bar{\iota}$-sequences in D', namely:

$$I(\bar{\iota})' = \{(z', \iota) : (z', \iota) \in \rho'_{\bar{i}}\} = \{(z'_1, \iota), ..., (z'_n, \iota)\}.$$

Since D' is minimal, no (z'_j, ι) in $I(\bar{\iota})'$ is in a shaded zone. Create a diagram, D'' say, from D' by swapping $I(\bar{\iota})'$ for the i-sequence $(Z\backslash(ShZ \cup \{z'_1, ..., z'_n\}), \iota)$. The diagram D'' represents its constants simply and is semantically equivalent to D_{min}. Again, it can be shown that the i-sequence $(Z\backslash(ShZ \cup \{z'_1, ..., z'_n\}), \iota)$ is in D_{min}. Then, since D_{min} represents its constants simply, it follows that $(Z\backslash(ShZ \cup \{z'_1, ..., z'_n\}), \iota) = (r, \iota)$, that is $\{z'_1, ..., z'_n\} = r$. Since D_{min} was created using algorithm 1, we know that

$$|Z\backslash(ShZ \cup r)| \geq 2|r| - 1.$$

Moreover, since D' is minimally cluttered,

$$|Z\backslash(ShZ \cup r)| \leq 2|r| - 1.$$

Therefore,

$$|Z\backslash(ShZ \cup r)| = 2|r| - 1.$$

This implies that we can swap $I(\bar{\iota})'$ in D' for (r, ι) without changing the clutter score.

The proof that case (c) cannot happen is similar to part 1 case (c). To finish part 2, we create a new minimally cluttered diagram,

$$D'_2 = (L, Z, ShZ, \rho'_{\otimes 2}, \rho'_{i2}, \rho'_{\bar{i}2})$$

obtained from D' by swapping all of the $I(\bar{\iota})'$ for which there exists an (r, ι) in D_{min}, for $(Z\backslash(ShZ \cup r), \iota)$. Importantly, As a result of this step, the i-sequences in D'_2 match those in D_{min}, that is we have

$$\rho'_{i2} = \rho_i.$$

.

Part 3 We now show that the $\bar{i}$-sequences in D'_2 match those in D_{min}, that is we have $\rho'_{\bar{i}2} = \rho_{\bar{i}}$. Let (z'_2, ι) be in $\rho'_{\bar{i}2}$. Then either:

(a) there is a region, r, such that (r, ι) is in D_{min},
(b) there is a zone, z, such that (z, ι) is in D_{min}, or
(c) neither of the above.

We show that neither case (a) nor case (c) can happen. In case (a), we also have (r, ι) in D'_2, since the i-sequences in D'_2 match those in D_{min}. Therefore, both (z'_2, ι) and (r, ι) are in D'_2. However, since D'_2 is minimally cluttered we know, by lemma 3, that D'_2 represents its constants simply. Therefore it cannot be that both (z'_2, ι) and (r, ι) are in D'_2. Hence case (a) cannot happen.

For case (c), we can choose a canonical interpretation for D_{min} where $\psi(\iota) = z'_2$ which models D_{min} but, trivially, causes the $\bar{i}$-sequence condition to fail for

D_2'. Since D_{min} and D_2' are semantically equivalent, this is a contradiction. Hence case (c) cannot happen.

Therefore, only case (b) holds. Lemma 6 and the semantic equivalence of D_2' and D_{min} imply that

$$\{(z'', \iota) : (z'', \iota) \in \rho_{\bar{i}2}\} = \{(z'', \iota) : (z'', \iota) \in \rho_{\bar{i}}\}.$$

Since (z_2', ι) was an arbitrary element of $\rho_{\bar{i}2}'$, we conclude that $\rho_{\bar{i}2}' \subseteq \rho_{\bar{i}}$. The proof that $\rho_{\bar{i}} \subseteq \rho_{\bar{i}2}'$ is similar. Hence

$$\rho_{\bar{i}} = \rho_{\bar{i}2}'.$$

Part 4. We have established, so far, that D_2' has the same i-sequences and the same $\bar{i}$-sequences as D_{min}. All that remains is to show that the $\otimes$-sequences match. Let $(r_2', \otimes)$ be an $\otimes$-sequence in D_2'. We show that none of the following occur in D_{min}:

(a) there is a region, r, such that $(r, \otimes)$ is in D_{min} and $r \subset r_2'$,
(b) there is a region, r, and a constant symbol, ι, such that (r, ι) is in D_{min} and $r \subseteq r_2'$,
(c) there is a region, r, and a constant symbol, ι, such that $r \subseteq r_2'$ and for each non-shaded zone, z, in D_{min} that is not in r (i.e. $z \in Z \backslash (ShZ \cup r)$) the $\bar{\iota}$-sequence (z, ι) is in D_{min}.

Suppose (a) occurs and let $(r, \otimes)$ be such an $\otimes$-sequence. Then in any model for D_{min}, $\Psi(r) \neq \emptyset$. Therefore, in any model for D_2', $\Psi(r) \neq \emptyset$. So, we can place the $\otimes$-sequence $(r, \otimes)$ in D_2' without changing its models. That is $D_2' + (r, \otimes)$ is semantically equivalent to D_2'. Then $(r_2', \otimes)$ is redundant from $D_2' + (r, \otimes)$ and $D_2' + (r, \otimes) - (r_2', \otimes)$ has a lower clutter score than D_2'. This contradicts the minimality of D_2', so case (a) cannot happen.

Suppose (b) occurs and let (r, ι) be such an i-sequence. Then (r, ι) is also in D_2', since the i-sequences match those of D_{min}. But then $(r_2', \otimes)$ is redundant in D_2', contracting its minimality, so case (b) cannot happen.

Suppose case (c) occurs and let r and ι be such a region and constant symbol respectively. Then again we can show that $(r_2', \otimes)$ is redundant in D_2', contracting its minimality. Hence, case (c) cannot happen.

We now show that $(r_2', \otimes)$ is in D_{min}. Suppose that $(r_2', \otimes)$ is not in D_{min}. There are three key points to make:

(d) The assumption that $(r_2', \otimes)$ is not in D_{min} together with the fact that case (a) cannot occur implies that for every $\otimes$-sequence, $(r, \otimes)$ in D_{min}, we have $r \not\subseteq r_2'$, so there is a zone, z, such that $z \in r$ and $z \notin r_2'$. For each $\otimes$-sequence in D_{min} choose such a zone and denote the set of chosen zones $CZ(\otimes)$.
(e) The fact that case (b) cannot occur implies that for every i-sequence, (r, ι) in D_{min}, we have $r \not\subseteq r_2'$, so there is a zone, z, such that $z \in r$ and $z \notin r_2'$. For each i-sequence in D_{min} choose such a zone and denote the set of chosen zones $CZ(i)$.

(f) The fact that case (c) cannot occur implies that for $\bar{i}$-sequence,(z, ι), in D_{min} there is a zone, z, that is not in r_2' and is not in $\{z' : (z', \iota) \in \rho_{\bar{i}}\}$. For each $\{z' : (z', \iota) \in \rho_{\bar{i}}\}$ choose such a zone and denote the set of chosen zones $CZ(\bar{i})$.

We now construct a canonical interpretation, $\mathcal{I} = (U, \psi, \Psi)$ for D_{min}, remove all of the zones in r_2' from its universal set and show that the result is still a model for D_{min} but is not a model for D_2'. The only choices to be made when constructing a canonical interpretation are when defining ψ. We define ψ as follows:

(g) for each (r, ι) in D_{min}, define $\psi(\iota)$ to be the zone chosen in case (e),
(h) for each $\{z' : (z', \iota) \in \rho_{\bar{i}}\}$, define $\psi(\iota)$ to be the zone chosen in case (f),
(i) for all other constant symbols, choose an arbitrary zone in $CZ(\otimes) \cup CZ(i) \cup CZ(\bar{i})$.

Since $\mathcal{I}$ is a canonical interpretation for D_{min}, it is a model for D_{min}, by lemma 2. Now, we can remove all of the zones in r_2' from U to create a new interpretation, $\mathcal{I}' = (U \backslash r_2', \psi, \Psi')$, where:

(j) we observe that ψ is still well-defined, since all of the zones to which constant symbols map are in the reduced universal set, and
(k) the function Ψ' is obtained by restricting the codomain of Ψ to $\mathbb{P}(U \backslash r_2')$.

It is easy to show that $\mathcal{I}'$ models D_{min}. However, in $\mathcal{I}'$, we have $\Psi(r_2') = \emptyset$, so the $\otimes$-sequence condition fails for D_2'. Therefore, $\mathcal{I}'$ does not model D_2', which is a contradiction. Hence the assumption that $(r_2', \otimes)$ is not in D_{min} is false. Therefore $\rho'_{\otimes 2} \subseteq \rho_{\otimes}$.

We now show that $\rho_{\otimes} \subseteq \rho'_{\otimes 2}$. Let $(r, \otimes)$ be an $\otimes$-sequence in D_{min}. We show that none of the following occur in D_2':

(l) there is a region, r_2', such that $(r_2', \otimes)$ is in D_2' and $r_2' \subset r$,
(m) there is a region, r_2', and a constant symbol, ι, such that (r_2', ι) is in D_2' and $r_2' \subseteq r$,
(n) there is a region, r_2', and a constant symbol, ι, such that $r_2' \subseteq r$ and for each non-shaded zone, z, in D_2' that is not in r_2' (i.e. $z \in Z \backslash (ShZ \cup r)$) the $\bar{\iota}$-sequence (z, ι) is in D_2'.

Suppose (l) occurs and let $(r_2', \otimes)$ be such an $\otimes$-sequence. Then, since we have already shown that $\rho'_{i2} \subseteq \rho_i$, we know that $(r_2', \otimes)$ is in D_{min}. But then $(r, \otimes)$ is redundant in D_{min}, and would have been removed by algorithm 1 at step 4, so case (l) cannot happen.

Suppose (m) occurs and let (r_2', ι) be such an i-sequence. Then (r_2', ι) is also in D_{min}, since the i-sequences match those of D_2'. But then $(r, \otimes)$ is redundant in D_{min}, and would have been removed by algorithm 1 at step 4, so case (m) cannot happen.

Suppose case (n) occurs and let r and ι be such a region and constant symbol respectively. Then again we can show that $(r, \otimes)$ is redundant in D_{min}, and would have been removed by algorithm 1 at step 4. Hence, case (n) cannot happen.

We now show that $(r, \otimes)$ is in D_2'. Again, we can construct a model for D_2' that is not a model for D_{min}, using the same approach as above, this time using cases (m) and (n). Hence the assumption that $(r, \otimes)$ is not in D_2' is false. Hence $\rho_i \subseteq \rho_{i2}'$ and we have $\rho_i = \rho_{i2}'$.

Having established that the $\otimes$-sequences, the i-sequences and $\bar{i}$-sequences are the same in D_{min} and D_2', we conclude that $D_{min} = D_2'$. Therefore D_{min} is minimally cluttered. $\square$